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A Jacobson Radical for Hopf Module Algebras*

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INTRODUCTION

Let H be a bialgebra over the commutative associative ring K with unit. This paper examines the concept of an H-radical for (associative) H-module algebras (also called algebras over H), based on the Amitsur–Kurosh general radical theory (Definitions 2, 3, Propositions 1–5, below). In particular, a Jacobson-type H-radical $\mathscr{J}$ is constructed as the upper H-radical generated by the left H-primitive H-module algebras (Definition 3, Theorem 1). Another H-radical of interest is J_H, which consists of all associative H-module algebras whose underlying algebra is in J, the ordinary Jacobson radical for associative K-algebras (Propositions 2, 3).

The main theorems on $\mathscr{J}$ are in Section 2, where we show that if H is irreducible (also called filtered), and if H is a flat K-module, then (Theorem 2) for any H-module algebra A, $\mathscr{J}(A)$ is equal to the intersection of all left H-primitive ideals of A; (Theorem 3 and Corollary) $\mathscr{J}$ is a strongly hereditary H-radical; (Theorem 4) $\mathscr{J}(A) = J(A \# H) \cap A$, where $A \# H$ is the smash product of A with H; (Corollary 1 to Theorem 4) $\mathscr{J}(A)$ is the intersection of all right H-primitive ideals of A; (Theorem 5) $\mathscr{J}(A)$ contains all the left or right H-ideals of A which are in $\mathscr{J}$; (Theorem 6) $\mathscr{J} \subseteq J_H$. An example is then provided showing that it is possible to have $\mathscr{J}(A) \neq J_H(A)$ for a non-Artinian H-module algebra A, whereas Theorem 7 shows that $\mathscr{J}(A) = J_H(A)$ if A is (left or right) Artinian.

An example motivating this study is the case in which A is an (associative) K-algebra and H is the universal enveloping algebra of the Lie algebra of derivations of A.

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1. Basic Definitions and Radical Constructions

Throughout this paper K will denote a commutative associative ring with unit. Algebras, bialgebras, and tensor products over base ring K are considered. The reader is referred to [6, p. 53] for the definition of a bialgebra H over K, and to [6, p. 153] for the definition of an H-module algebra, except that we do not assume that H-module algebras are necessarily unital. Reference [6] defines these concepts in the case that K is a field, but the same definitions (as well as that of a coalgebra over K) make sense in the general case considered here. For this general approach we will use results from [3, Section 1]. To be explicit, A is an H-module algebra if A is a K-algebra which is an H-module with the measuring condition written out as follows. If μ: $H \otimes A \to A$ is the measuring of A by H (or action of H on A), we will also write $\mu(h \otimes a) = h \cdot a$ so that the measuring condition reads $h \cdot (ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for all $a, b \in A$, $h \in H$. For more description of the summation notation in the last statement see [6, p. 10]. It is assumed that $1_H \cdot a = a$ for all $a \in A$, where 1_H is the unit of H. The measuring is called *unital* if A has a unit element 1 and if $h \cdot 1 = \epsilon(h)1$ for all $h \in H$, where ϵ is the counit of H. Note that H-module algebras are the multiplicative objects in the monoidal category of H-modules.

Let $\mathscr{H}$ be the category of all associative H-module algebras, where H is a given bialgebra over K. The objects of $\mathscr{H}$ are all associative H-module algebras. The morphisms of $\mathscr{H}$ are those algebra homomorphisms φ: $A \to B$, $A, B \in \mathscr{H}$, which are also H-module maps. Such a φ will be called an *H-homomorphism*. An ideal I of an H-module algebra A is called an *H-ideal* if the action of H on A leaves I invariant. An H-ideal is the same thing as the kernel of an H-homomorphism. In particular, if I is an H-ideal of A, then I is the kernel of the natural H-homomorphism $A \to A/I$, where A/I is an H-module algebra via $h \cdot (a + I) = (h \cdot a) + I$ for all $h \in H$, $a \in A$. The sum and intersection of H-ideals are H ideals. The image $\varphi(A)$ of an H-module algebra A by an H-homomorphism φ is naturally H-isomorphic to A/I for the H-ideal $I = \ker \varphi$.

The concept of a module for an A in the category $\mathscr{H}$ is made explicit by means of the following definition.

Definition 1. Suppose A is an H-module algebra and M is a left A-module. Then M is a left *A, H-module* provided M is also a unital left H-module (where H is thought of as an algebra), and

$$h(am) = \sum_{(h)} (h_{(1)} \cdot a)\, h_{(2)}(m)$$

for all $h \in H$, $a \in A$, $m \in M$, where $\Delta h = \sum_{(h)} h_{(1)} \otimes h_{(2)}$. If A has a unit, then the A, H-module M is called *unital* if M is unital as a left A-module. Note that A, H-modules are the multiplicative actions in the monoidal category of H-modules. The A, H-module M is *irreducible* if $AM \neq 0$ and M has no proper nonzero A, H-submodule (i.e., no K-subspace closed under action by A and H); in addition, if A has a unit then it is further required that M be unital. An H-module algebra A is left *H-primitive* provided A has a left, A, H-module M which is faithful as an A-module, and irreducible as an A, H-module.

Suppose A is an H-module algebra. The *smash product* (or *semidirect product*) $A \# H$ of A by H is the associative algebra consisting of the elements of $A \otimes H$ ($a \otimes h$ written $a \# h$) with products defined by

$$(a \# g)(b \# h) = \sum_{(g)} a(g_{(1)} \cdot b) \# g_{(2)}h.$$

If A has a unit 1_A and the measuring of H on A is unital, then $1_A \# 1_H$ is a unit for $A \# H$.

LEMMA 1. (i) *If A is an H-module algebra such that either A does not have a unit, or A does have a unit but the measuring is not unital, then one can adjoin a unit to A to obtain an H-module algebra $A_1 = A + K$ (direct as K-spaces) for which the measuring is unital, where the action of H on A is defined by*

$$h \cdot (a + k) = h \cdot a + \epsilon(h)k,$$

for all $h \in H$, $a \in A$, $k \in K$. A is then embedded as an H-ideal in A_1 in the natural fashion. If M is an A, H-module, then M is a unital A_1, H-module under the action $(a + k)m = am + km$ for all $a \in A$, $k \in K$, $m \in M$.

(ii) *If M is an A, H-module, then M is an $A \# H$-module under the action $(a \# h)m = ah(m)$ for all $a \in A$, $h \in H$, $m \in M$. If M is an irreducible left A, H-module, then M is an irreducible left $A \# H$-module.*

(iii) *If the measuring of H on A is unital, and if M is an (irreducible) left $A \# H$-module, then M is an (irreducible) left A, H-module under the action $am = (a \# 1_H)m$, $h(m) = (1_A \# h)m$ for all $a \in A$, $h \in H$, $m \in M$.*

Proof. The details of the proof are mostly straight-forward, being based on definitions. However, the last statement in (ii) needs comment. As stated, A need not have a unit. If A does have a unit, then the proof is easy. Assume then that A does not have a unit. As in (i) adjoin a unit to A to get $A_1 = A + K$. M is then an irreducible A_1, H-module and an irreducible $A_1 \# H$-module. Since A is a direct summand (as a K-space) of A_1, $A \# H$ is embedded in $A_1 \# H$ in the natural fashion. Let $S = \{m \in M\colon (A \# H)m = 0\}$. S is an $A_1 \# H$-submodule of M, hence $S = M$ or $S = 0$. If $S = M$, then $AM = 0$, contrary to hypothesis. Thus $S = 0$. So for any nonzero $m \in M$, $(A \# H)m = M$. Now suppose N is a nonzero $A \# H$-submodule of M. Then $(A \# H)N \subseteq N$ and $(A \# H)N \supseteq (A \# H)n = M$ for any nonzero $n \in N$, thus $N = M$. It has been shown that M is an irreducible *$A \# H$-module.*

DEFINITION 2. A nonempty subset $\mathscr{R}$ of $\mathscr{H}$ is an *H-radical* provided

(a) If $A \in \mathscr{R}$, then $\varphi(A) \in \mathscr{R}$ for every H-homomorphism φ of A.

(b) If $A \in \mathscr{H}$, $A \notin \mathscr{R}$, then there exists a nonzero H-homomorphism φ of A such that $\varphi(A)$ has no nonzero H-ideals in $\mathscr{R}$.

The following notation will be useful. For $\chi \subseteq \mathscr{H}$,

$\mathfrak{S}(\chi) = \{A \in \mathscr{H} : A$ has no nonzero H-ideal in $\chi\}$,

$\mathfrak{R}(\chi) = \{A \in \mathscr{H}; A$ has no nonzero H-homomorphic image in $\chi\}$.

Given an H-radical $\mathscr{R}$ and $A \in \mathscr{H}$, A is said to be *$\mathscr{R}$-radical* provided $A \in \mathscr{R}$ and A is said to be *$\mathscr{R}$-semisimple* provided $A \in \mathfrak{S}(\mathscr{R})$. The H-ideal

$$\mathscr{R}(A) = \sum \{I: I \text{ is an } H\text{-ideal of } A, \text{ and } I \in \mathscr{R}\}$$

is called the *$\mathscr{R}$-radical of A*. For each $A \in \mathscr{H}$, $\mathscr{R}(A) \in \mathscr{R}$, $A/\mathscr{R}(A) \in \mathfrak{S}(\mathscr{R})$, and $\mathscr{R}(A) = \bigcap \{I : I$ is an H-ideal of A, and $A/I \in \mathfrak{S}(\mathscr{R})\}$.

PROPOSITION 1. *Suppose $\mathscr{S} \subseteq \mathscr{H}$ satisfies the following condition: $A \in \mathscr{S}$ implies every nonzero H-ideal of A has a nonzero H-homomorphic image in $\mathscr{S}$. Then* (i) *$\mathfrak{R}(\mathscr{S})$ is an H-radical*; (ii) *$\mathfrak{S}(\mathfrak{R}(\mathscr{S}))$ is the minimal semisimple class in $\mathscr{H}$ containing $\mathscr{S}$*; (iii) *if $\mathscr{R}$ is an H-radical for which $\mathfrak{S}(\mathscr{R}) \supseteq \mathscr{S}$, then $\mathscr{R} \subseteq \mathfrak{R}(\mathscr{S})$.*

Because of (iii) $\mathfrak{R}(\mathscr{S})$ is called the *upper H-radical generated by $\mathscr{S}$*. Generally, the proofs of the propositions in this section are similar to known proofs in general radical theory, or are otherwise straightforward. In particular, the proof of Proposition 1 resembles [4, Lemma 3, p. 6].

We wish to apply Proposition 1 to the class $\mathscr{S}$ of all left H-primitive H-module algebras. The condition in the hypothesis of Proposition 2 is verified in Theorem 1 below.

THEOREM 1. *If A is left H-primitive and I is a nonzero H-ideal of A, then I is left H-primitive.*

Proof. Suppose M is an irreducible left A, H-module which is faithful as an A-module. Then M is an I, H-module which is faithful as an I-module. Suppose N is an I, H-submodule of M. Let C denote the K-subspace of $A \# H$ generated by $\{x \# h: x \in I, h \in H\}$. Then C is an ideal of $A \# H$.

Thus CN is an $A \# H$-submodule of M, whereas M is an irreducible $A \# H$-module, using Lemma 1(ii). Hence $CN = 0$ or $CN = M$. If $CN = M$, then $N = M$, and the proof is completed. On the other hand, if $CN = 0$, let $S = \{m \in M,\ Cm = 0\}$. Now $S \supseteq N$ and S is an $A \# H$-submodule of M, so $S = 0$ or $S = M$. If $S = 0$, then $N = 0$, and the proof is again completed. If $S = M$, then $CM = 0$, so $IM = 0$, which would imply $I = 0$ since M is faithful as an A-module. This case is therefore not possible and so I is left H-primitive.

Theorem 1 justifies the use of Proposition 1 to form the upper H-radical $\mathscr{J} = \mathfrak{R}(\mathscr{S})$, where $\mathscr{S}$ is the class of all left H-primitive H-module algebras. More can be proved about $\mathscr{J}$, and this is done in Section 2, if one assumes further conditions on H. The conditions of interest in this paper are stated explicitly, and explained at the end of this section and the beginning of Section 2.

Another H-radical of interest is obtained from the (ordinary) Jacobson radical J for associative rings (or K-algebras). The general procedure is spelled out in the following proposition.

PROPOSITION 2. *Assume that ρ is an ordinary radical for associative K-algebras. Then ρ_H, the class of all H-module algebras whose underlying algebra is in ρ, is an H-radical.*

Hence J_H is an H-radical. Section 2 gives the relationships, under appropriate conditions, among $\mathscr{J}(A)$, $J_H(A)$, and $J(A \# H)$, where A is an H-module algebra. Structure theorems for $A/J_H(A)$ where $H = \mathscr{U}(\text{der } A)$, with certain finiteness conditions, can be found in [1, p. 452].

DEFINITION 3. The H-radical $\mathscr{R}$ is a *hereditary* H-radical provided $A \in \mathscr{R}$ implies $I \in \mathscr{R}$ for every H-ideal I of A. $\mathscr{R}$ is *strongly hereditary* provided $\mathscr{R}(I) = \mathscr{R}(A) \cap I$ for every H-ideal I of A.

As usual, every strongly hereditary H-radical is hereditary. If $\mathscr{R}$ is a hereditary H-radical then $\mathscr{R}(A) \cap I \subseteq \mathscr{R}(I)$ for all H-ideals I of A. If $\mathscr{R}$ is a strongly hereditary H-radical and $A \in \mathfrak{S}(\mathscr{R})$, then $I \in \mathfrak{S}(R)$ for every H-ideal I of A.

PROPOSITION 3. *Suppose ρ is an ordinary radical for associative K-algebras. If ρ is hereditary, then ρ_H is a hereditary H-radical and*

$$\rho_H(A) = \sum \{I\colon I \textit{ is an } H\textit{-ideal of } A, \textit{ and } I \subseteq \rho(A)\}.$$

If ρ is strongly hereditary, then ρ_H is a strongly hereditary H-radical.

As an immediate application of Proposition 3, one gets that J_H is a strongly hereditary H-radical.

PROPOSITION 4. *Suppose $\mathscr{R}$ is a hereditary H-radical such that all $A \in \mathscr{H}$ with $A^2 = 0$ are in $\mathscr{R}$. Then $\mathscr{R}$ is a strongly hereditary H-radical.*

Proof. Let I be an H-ideal of $A \in \mathscr{H}$. It suffices to show that $\mathscr{R}(I)$ is an H-ideal of A. Set $R = \mathscr{R}(I)$. Then $(AR + R)^2 \subseteq R$ and hence $(AR + R)/R \in \mathscr{R}$ and $(AR + R)/R$ is an H-ideal of $I/R \in \mathfrak{S}(\mathscr{R})$. Hence $(AR + R)/R = 0$, i.e., $AR \subseteq R$. Similarly $RA \subseteq R$ and thus $R = \mathscr{R}(I)$ is an H-ideal of A.

For ordinary radical theory, Proposition 4 can be proved without the assumption that $\mathscr{R}$ contains all A such that $A^2 = 0$. Whether or not this assumption can be deleted for hereditary H-radicals is left open in this paper. In Section 2, Proposition 4 will be applied to show that $\mathscr{J}$ is a strongly hereditary H-radical.

PROPOSITION 5. *Suppose $\mathscr{S} \subseteq \mathscr{H}$ satisfies the condition in the hypothesis of Proposition* 1, *and let $\mathscr{R} = \mathfrak{R}(\mathscr{S})$. Suppose further that for all $A \in \mathscr{H}$:* (i) *If I is a nonzero H-ideal of A and $I \in \mathscr{S}$, then there exists an H-ideal B of A such that $A/B \in \mathscr{S}$ and $I \not\subseteq B$.* (ii) $A^2 = 0$ *implies $A \notin \mathscr{S}$. Then, for every $A \in \mathscr{H}$,*

$$\mathscr{R}(A) = \bigcap \{I\colon I \text{ is an } H\text{-ideal of } A, \text{ and } A/I \in \mathscr{S}\}.$$

A similar result is that if ρ is a hereditary ordinary upper radical generated by a class σ, and if $\rho(A) = \bigcap \{I : I$ is an ideal of A, and $A/I \in \sigma\}$, then $\rho_H(A) = \bigcap \{I_H : A/I \in \sigma\}$, where I_H is the sum of all the H-ideals of A contained in I. This was used for J_H in [1, p. 452], using $H = \mathscr{U}(\operatorname{der} A)$, σ equal to the set of primitive associative rings.

Proposition 5 resembles [4, Lemma 80, p. 139], which is concerned with the topic of special radicals for associative rings. Under appropriate conditions, Proposition 5 will be applied to $\mathscr{J} = \mathfrak{R}(\mathscr{S})$, $\mathscr{S}$ the left H-primitive H-module algebras, in Theorem 2 in Section 2.

DEFINITION 4. H is an *irreducible* bialgebra over K provided there exists a denumerable sequence of K-subspaces H_i of H, $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots \subseteq H$ where $H_0 = K1_H$, $H = \bigcup H_i$, $H_iH_j \subseteq H_{i+j}$ and $\Delta H_n \subseteq \sum_{i=0}^{n} \operatorname{Im}(H_i \otimes H_{n-i})$. Here $\operatorname{Im}(H_i \otimes H_{n-i})$ denotes the image of the canonical map of $H_i \otimes H_{n-i}$ into $H \otimes H$.

This definition of irreducible bialgebra is the same as that of filtered bialgebra in [3, p. 10], and is equivalent to H being irreducible as a coalgebra in the sense of [7] where K is a field.

The following are important examples of irreducible bialgebras over K.

(1) K a field and H irreducible as a coalgebra.

(2) K not necessarily a field, but H generated as an algebra by $P(H) = \{h \in H : \Delta h = h \otimes 1 + 1 \otimes h\}$, the "primitive" elements of H. One easily checks in this case that a filtration is provided by setting $H_n = \sum_{i=0}^{n} P(H)^i$, $n = 0, 1, 2,\ldots$, where, by convention $P(H)^0 = K1_H$.

(3) $H = \mathscr{U}(L)$, the universal enveloping algebra of the Lie algebra L. This is a special case of (2).

If $\{H_i\}$ is a filtration of the bialgebra H over K, and if we set $H_n^+ = H_n \cap (\ker \epsilon)$, $H^+ = H \cap (\ker \epsilon)$, then one has the following decompositions:

$$H = H^+ \oplus K1_H, \qquad H^+ = \bigcup_{i=0}^{\infty} H_i^+,$$

where the sum is direct as K-spaces. As is shown in [3, p. 10], if H is irreducible and $h \in H_n^+$, then $\Delta h = h \otimes 1 + 1 \otimes h + y$ for some $y \in \sum_{i=1}^{n-1} \operatorname{Im}(H_i \otimes H_{n-i})$.

Lemma 2 below proves one fact about irreducible bialgebras that will be useful in Section 2.

LEMMA 2. *Assume that H is irreducible and that A is an H-module algebra. Then the annihilator in A of a left A, H-module M is an H-ideal of A.*

Proof. Let $I = \{a \in A : aM = 0\}$, an ideal of A. It needs to be shown that $a \in I$ implies $h \cdot a \in I$ for all $h \in H$. Writing, as above $H = H^+ + K1_H$, one can assume $h \in H^+ = \bigcup_{i=0}^{\infty} H_i^+$. This makes h an element of some H_n^+. If $n = 0$, then $h = 0$ and $h \cdot a = 0 \cdot a = 0$ is in I. The induction assumption is that $g \cdot a$ is in I for all g in H_i^+ and for all i less than n. Since H is irreducible one can write

$$\Delta h = h \otimes 1 + 1 \otimes h + \sum g_i \otimes f_i,$$

where g_i, f_i belong to subspaces of index less than n. Then for any $m \in M$,

$$\begin{aligned}(h \cdot a)m &= h(am) - ah(m) - \sum (g_i \cdot a) f_i(m) \\ &= 0 - 0 - 0 = 0,\end{aligned}$$

since $aM = 0$ and $g_i \cdot a \in I$ for all i by the induction assumption. Therefore, $h \cdot a \in I$, as claimed.

As a slight generalization note that essentially the same argument shows that if N is an A, H-submodule of M then $\{a \in M : aM \subseteq N\}$ is also an H-ideal of A. Also, analogous right-handed versions for the above are true.

2. Main Theorems on $\mathscr{J}$, J_H, $J(A \# H)$

The following two basic assumptions on the bialgebra H over K occur frequently in this section:

(1) H is irreducible,

(2) H is a flat K-module.

For example, if K is a field then H is flat; if $K = Z$ then H is flat if and only if H is torsion-free. As a consequence, if A is an associative H-module algebra, and if S is an H-invariant subalgebra of A, then $S \# H$ is embedded injectively in $A \# H$. Therefore, if I is an H-ideal of A, one can naturally consider $I \# H$ as an ideal of $A \# H$, and it is for the sake of this type of application that we assume that H is flat.

Theorem 2 states that $\mathscr{J}(A)$ is the intersection of the left H-primitive ideals of A. As expected, an H-ideal P of A is defined to be a *left H-primitive ideal* provided A/P is left H-primitive; i.e., P is the annihilator in A of an irreducible left A, H-module. (Lemma 2 shows immediately that every such annihilator is an H-ideal of A.)

Theorem 2. *Assume that H is an irreducible bialgebra over K, and that H is a flat K-module. Then for an H-module algebra A,*

$$\mathscr{J}(A) = \bigcap \{P\colon P \text{ is an } H\text{-ideal of } A \text{ and } A/P \text{ is left } H\text{-primitive}\}.$$

The following lemma establishes one of the sufficient conditions (see Proposition 5).

Lemma 3. *Assume the hypotheses on H in the statement of Theorem 2. Suppose I is a nonzero H-ideal of the H-module algebra A, and that I is itself a left H-primitive H-module algebra. Then there exists an H-ideal B of A such that A/B is left H-primitive and $I \not\subseteq B$.*

Proof. Let M be an irreducible left I, H-module, faithful as an I-module. Then M is an irreducible left $I \# H$-module by Lemma 1(ii). As in the proof of Lemma 1(ii), one has that for any nonzero $n \in M$, $(I \# H)n = M$. Work with some such fixed generator n. Since H is a flat K-module, consider $I \# H$ as an ideal of $A \# H$ and make M into an $A \# H$-module by defining $u(vn) = (uv)n$ for all $u \in A \# H$, $v \in I \# H$. To show that this action is well-defined it must be shown that if $vn = 0$, then $(uv)n = 0$. One has the conventional calculation, assuming $vn = 0$:

$$(I \# H)((uv)n) = ((I \# H)(uv))n = ((I \# H)u)(vn) = 0.$$

As before, anything annihilated by $I \# H$ is zero, so $(uv)n = 0$. Now make M into an A-module by setting $am = (a \# 1)m$ for any $m \in M$, $a \in A$. If a is in I, then ("new action") $am = (a \# 1)m = am$ ("old action"), giving the correct module action of I on M. Thus any A-submodule of M is an I-submodule of M. Once it is shown that M is an A, H-module, it follows that M is an irreducible A, H-module. So we claim that

$$h(am) = \sum_{(h)} (h_{(1)} \cdot a)\, h_{(2)}(m) \tag{†}$$

for all $h \in H$, $a \in A$, $m \in M$. But

$$\begin{aligned} am = (a \# 1)m &= (a \# 1)\left[\left(\sum_{i=1}^{k} x_i \# g_i\right) n\right] \\ &= \sum_{i=1}^{k} (a \# 1)[(x_i \# g_i)n] \end{aligned}$$

for some $x_i \in I$, $g_i \in H$ and

$$h(am) = \sum_{i=1}^{k} h((a \# 1)[x_i \# g_i)n]).$$

So one needs show (†) when m has the form $m = (x \# g)n$, $x \in I$. We have

$$\begin{aligned} h(am) &= h((a \# 1)[(x \# g)n]) = h((ax \# g)n) \\ &= h((ax)\, g(n)) \qquad \text{since} \quad ax \in I \\ &= \sum_{(h)} (h_{(1)} \cdot (ax))\, h_{(2)} g(n) \\ &= \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot x)\, h_{(3)} g(n), \end{aligned}$$

where here and below the fact that $\sum_{(h)} \Delta h_{(1)} \otimes h_{(2)} = \sum_{(h)} h_{(1)} \otimes \Delta h_{(2)}$ (i.e., coassociativity) is used, which justifies the use of three subscripts as displayed. On the other hand,

$$\begin{aligned} \sum_{(h)} (h_{(1)} \cdot a)\, h_{(2)}(m) &= \sum_{(h)} (h_{(1)} \cdot a)\, h_{(2)}((x \# g)n) \\ &= \sum_{(h)} (h_{(1)} \cdot a)\, h_{(2)}(xg(n)) \\ &= \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot x)\, h_{(3)} g(n) \end{aligned}$$

and the two end results are equal. So M is indeed an A, H-module.

Now let $B = \{a \in A : aM = 0\}$. B is an H-ideal of A since H is irreducible, by Lemma 2. It must be the case that $B \cap I = 0$ since $BM = 0$ and M is a faithful I-module (in fact $B = \{a \in A : aI = 0\}$). Also, M is a faithful A/B-module, and an irreducible A/B, H-module via $(a + B)m = am$. Thus A/B is left H-primitive, and $I \not\subseteq B$. This finishes the lemma.

Proof of Theorem 2. In addition to Lemma 3 all one needs to observe is that if A is an H-module algebra with $A^2 = 0$, then A cannot be left H-primitive, since AM would be zero whenever M was an irreducible A, H-module, a contradiction. Proposition 5 is now applied to finish the proof.

An H-module algebra A is *H-simple* provided the only H-ideals of A are 0 and A, and $A^2 \neq 0$.

COROLLARY. *An H-simple algebra A is $\mathscr{J}$-semisimple if and only if A is left H-primitive.*

Proof. If A is H-primitive then A is $\mathscr{J}$-semisimple. If A is H-simple and H-semisimple, then A has an irreducible left A, H-module M such that $AM \neq 0$. But the annihilator of M is an H-ideal of A, not equal to A, and hence is zero, so A is left H-primitive.

THEOREM 3. *Assume that H is an irreducible bialgebra over K, and that H is flat as a K-module. Then $\mathscr{J}$ is a hereditary H-radical.*

Proof. It must be shown that if $A \in \mathscr{J}$ and if I is an H-ideal of A, then $I \in \mathscr{J}$. By Theorem 2, $A \in \mathscr{J}$, i.e., $A = \mathscr{J}(A)$, means that A has no irreducible left A, H-modules. We show that I also has no irreducible left I, H-modules, in which case $I = \mathscr{J}(I)$, so $I \in \mathscr{J}$. Suppose that M is an irreducible left I, H-module, hence $IM \neq 0$. Lemma 3 shows how to make M into an irreducible left A, H-module such that the action of A on M when restricted to I gives the original action of I on M. (Here it is noted that to make M into an A, H-module requires only $IM \neq 0$, and not necessarily that M be a faithful I-module.) But by the assumption about A (that it has no irreducible left A, H-modules) this gives a contradiction. Therefore $I = \mathscr{J}(I)$ and $I \in \mathscr{J}$, so that $\mathscr{J}$ is hereditary.

COROLLARY. *$\mathscr{J}$ is a strongly hereditary H-radical. That is, for an H-module algebra A and an H-ideal I of A,*

$$\mathscr{J}(I) = \mathscr{J}(A) \cap I.$$

Proof. This follows from Theorem 3 and Proposition 4 since if A is an H-module algebra such that $A^2 = 0$, then $A \in \mathscr{J}$.

In what follows, regard A as a subalgebra of $A \# H$ via the canonical embedding $a \to a \# 1$.

THEOREM 4. *Assume that H is an irreducible bialgebra over K and that H is a flat K-module. Then $\mathcal{J}(A) = J(A \# H) \cap A$.*

Proof. The theorem can be proved if one first assumes that the measuring on A is unital (hence that A has a unit), and then remove this restriction. Assuming then that the measuring is unital, first observe that the irreducible left A, H-modules are exactly the irreducible left $A \# H$-modules by (ii) and (iii) of Lemma 1. Applying the constructions in (ii) and (iii), first one, then the other, preserves the module action with which one starts. Now $J(A \# H)$ is the intersection of the annihilators of irreducible left $A \# H$-modules, and $\mathcal{J}(A)$ is the intersection of the annihilators of irreducible left A, H-modules. These annihilators correspond as follows. Consider $P = \{a \in A : aM = 0\}$, where M is an irreducible left A, H-module. Consider M as an irreducible left $A \# H$-module as in (ii) of Lemma 1. Then $(P \# 1)M = P1(m) = PM = 0$. On the other hand, if $(a \# 1)M = 0$, then $a1(M) = aM = 0$, and so $a \in P$. So if $Q = \{u \in A \# H : uM = 0\}$, then $Q \cap A = P$. These arguments are reversible, using (iii) of Lemma 1 this time. Hence $\bigcap (Q \cap A) = \bigcap P$, where Q ranges over the left primitive ideals of $A \# H$, P ranges over the left H-primitive ideals of A. Hence $J(A \# H) \cap A = \mathcal{J}(A)$.

If the measuring is not unital, then use A_1, constructed in (i) of Lemma 1. Since $\mathcal{J}$ is a strongly hereditary H-radical and A is an H-ideal of A_1, we have

$$\begin{aligned}
\mathcal{J}(A) &= \mathcal{J}(A_1) \cap A \\
&= (J(A_1 \# H) \cap A_1) \cap A \text{ by the above,} \\
&= J(A_1 \# H) \cap A \\
&= (J(A_1 \# H) \cap A \# H) \cap A \\
&= J(A \# H) \cap A
\end{aligned}$$

since J is a strongly hereditary ordinary radical. This establishes the theorem in general.

COROLLARY 1. *$\mathcal{J}(A)$ is the intersection of all right H-primitive ideals of A.*

The definition of right H-primitive is the obvious one. The corollary follows easily from the theorem, since J is left or right definable, and makes $\mathcal{J}$ symmetrically definable as either the upper H-radical generated by the left H-primitive algebras or as the upper H-radical generated by the right H-primitive algebras.

COROLLARY 2. *If $A \in \mathscr{J}$, then $A \# H \in J$.*

Proof. If $A \in \mathscr{J}$, then in fact A does not have a unit, since otherwise $1_A \# 1_H \in J(A \# H)$, which is impossible. However, adjoin a unit to A as in (i) of Lemma 1, obtaining A_1, so that the following argument can be given. Since $A \in \mathscr{J}$, $A = \mathscr{J}(A) = J(A \# H) \cap A$, so that $A \# 1 \subseteq J(A \# H) = J(A_1 \# H) \cap (A \# H)$. Hence $A \# H = (A \# 1)(1 \# H) \subseteq J(A_1 \# H) \cap (A \# H) = J(A \# H)$ and so $A \# H = J(A \# H)$ or $A \# H \in J$.

THEOREM 5. *Assume that H is an irreducible bialgebra over K, and that H is a flat K-module. Then $\mathscr{J}(A)$ contains all the left (or right) H-ideals of A which are in $\mathscr{J}$.*

Proof. Assume $L \in \mathscr{J}$ is a left H-ideal of A. Then since $L \in \mathscr{J}$, $L \# H \in J$ by Corollary 2 above, and $L \# H$ is a left ideal of $A \# H$, hence $L \# H \subseteq J(A \# H)$. Thus, $L \# 1 \subseteq J(A \# H) \cap (A \# 1) = \mathscr{J}(A) \# 1$ and so $L \subseteq \mathscr{J}(A)$, as required. Similar argument applies to the right H-ideals.

The next theorem gives the general relationship between $\mathscr{J}$ and J_H.

THEOREM 6. *Assume that H is an irreducible bialgebra over K, and that H is a flat K-module. Then $J(A \# H) \cap A \subseteq J_H(A)$, hence $\mathscr{J} \subseteq J_H$ and $\mathscr{J}(A) \# H \subseteq J(A \# H)$.*

LEMMA 4. *Assume the hypotheses of Theorem 6. If S is an ideal of $A \# H$, then $S \cap A$ is an H-ideal of A.*

Proof. We show that if $a \# 1$ is in $S \cap (A \# 1)$, then $(h \cdot a) \# 1$ is in $S \cap (A \# 1)$ for all $h \in H$. Write $H = H^+ + K1_H$, $H^+ = \bigcup_{i=0}^{\infty} H_i^+$. It suffices to show $(h \cdot a) \# 1 \in S \cap (A \# 1)$ for all $h \in H^+$. If $h \in H_0^+ = K^+ = 0$, then $h = 0$ and $0 \# 1 = 0$ is in $S \cap (A \# 1)$. So assume the conclusion is true for all $g \in H_j^+$, for all j less than n, and let h be in H_n^+. Then

$$\Delta h = h \otimes 1 + 1 \otimes h + \sum g_i \otimes f_i,$$

where g_i, f_i belong to subspaces of index less than n. Then $(1 \# h)(a \# 1) = (h \cdot a) \# 1 + a \# h + \sum (g_i \cdot a) \# f_i$, which is in S since $a \# 1 \in S$ and S is an ideal of $A \# H$. Also, each $(g_i \cdot a) \# 1 \in S$ by the induction assumption, so $((g_i \cdot a) \# 1)(1 \# f_i) = (g_i \cdot a) \# f_i \in S$. Also $(a \# 1)(1 \# h) = a \# h \in S$. Going back to the original expression for $(1 \# h)(a \# 1)$, we get $(h \cdot a) \# 1 \in S \cap A$, finishing the lemma.

LEMMA 5. *Assume the hypotheses of Theorem 6. Suppose the measuring of H on A is unital. If $b \# 1$ is right invertible in $A \# H$, then b is right invertible in A.*

Proof. Suppose $b \# 1$ has right inverse $\sum_{i=1}^{n} c_i \# h_i$. Then $(b \# 1)(\sum c_i \# h_i) = 1 \# 1$ or $\sum bc_i \# h_i = 1 \# 1$. Now A is a unital $A \# H$-module via the basic action $(a \# h)x = a(h \cdot x)$, for all $a, x \in A$, $h \in H$. Apply both sides of $\sum bc_i \# h_i = 1 \# 1$ to 1_A. Then $\sum bc_i \epsilon(h_i) = 1_A$, i.e., $b(\sum c_i \epsilon(h_i)) = 1_A$, which shows b is right invertible in A.

Proof of Theorem 6. First assume that the measuring is unital. Since $J(A \# H)$ is an ideal of $A \# H$, Lemma 4 shows that $J(A \# H) \cap A$ is an H-ideal of A. Every element $a \# 1$ in $J(A \# H) \cap A$ is right-quasi-regular, that is $1 \# 1 + a \# 1 = (1 + a) \# 1$ has a right inverse in $A \# H$. Hence, by Lemma 5, $1 + a$ is right invertible in A. Thus $J(A \# H) \cap A$ is a right-quasi-regular H-ideal of A, and so is contained in $J_H(A)$. This proves the theorem when the measuring is unital. If the measuring is not unital, consider A_1. By the corollary to Theorem 3, $\mathscr{J}(A) = \mathscr{J}(A_1) \cap A \subseteq J_H(A_1) \cap A = J_H(A)$ since J_H is a strongly hereditary H-radical. This gives the theorem for general A. The other conclusions in Theorem 6 now follow easily.

The question naturally arises as to whether $J_H \subseteq \mathscr{J}$. It is shown in Theorem 7 below that this is the case for (left or right) Artinian algebras, but first a non-Artinian counterexample is given.

Example with $\mathscr{J}(A) \neq J_H(A)$. Let R denote the real field and let $A = R(x, y)$ be the algebra of all formal power series in commuting indeterminants x and y. Let $d = d/dx$ and let H be the bialgebra over R generated by d; a typical element of H is a finite polynomial in powers of d with coefficients in R. The Jacobson radical of A, $J(A)$, consists of all power series with zero constant term, and $J(A)$ contains the H-ideal B of A consisting of all power series of the form $p_0 + p_1 x + \cdots + p_i x + \cdots$, where each p_i is a power series in y with zero constant term. Now $B \subseteq J_H(A)$ and $y \in B$, so $y \in J_H(A)$. We propose to show $y \# d$ is *not* in $J(A \# H)$, verifying the example. The reason this will suffice is the following. If $J_H(A) = J_H(A) \# 1 \subseteq \mathscr{J}(A) = J(A \# H) \cap A$, then $J_H(A) \# H$ must also be contained in $J(A \# H)$, since the latter is an ideal of $A \# H$. But $y \# d \in J_H(A) \# H$ and $y \# d \notin J(A \# H)$, a contradiction.

Now every element u of $A \# H$ can be expressed in the form $u = \sum_{i=0}^{n} a_i \# d^i$, $a_i \in A$, for some nonnegative integer n. Suppose $y \# d$ were in $J(A \# H)$. Then $1 \# 1 + y \# d$ must have a left inverse u:

$$u(1 \# 1 + y \# d) = \left(\sum_{i=0}^{n} a_i \# d^i\right)(1 \# 1 + y \# d) = 1 \# 1.$$

This equation can be solved for the a_i by applying both sides to elements $1, x, x^2, \ldots,$ of A, since A is an $A \# H$-module. For example, applying both sides to 1, one gets $a_0 = 1$; applying both sides to x, one gets $a_0 x + a_0 y + a_1 = x$ or $a_1 = -y$, etc. Summing up,

$$u = 1 \# 1 - y \# d + y^2 \# d^2 - y^3 \# d^3 + \cdots \pm y^n \# d^n.$$

However,

$$\begin{aligned} u(1 \# 1 + y \# d) &= u + y \# d - y^2 \# d^2 + y^3 \# d^3 - \cdots \mp y^n \# d^n \\ &\quad \pm y^{n+1} \# d^{n+1} \\ &= 1 \# 1 \pm y^{n+1} \# d^{n+1} \\ &\neq 1 \# 1 \end{aligned}$$

since $y^{n+1} \# d^{n+1} \neq 0$. Hence one concludes that $A \# H$ does not contain a left-quasi-inverse for $y \# d$, and so $J_H(A) \not\subseteq \mathscr{J}(A)$.

THEOREM 7. *Assume that H is an irreducible bialgebra over K, and that H is a flat K-module. If $J_H(A)$ is nilpotent, then $J_H(A) = \mathscr{J}(A)$. Hence if A is (left or right) Artinian, then $J_H(A) = \mathscr{J}(A)$.*

Proof. This follows from the fact that if T is an H-ideal of A, then $(T \# H)^n \subseteq T^n \# H$. Applying this to $T = J_H(A)$, where $T^m = 0$ some m, one gets $(J_H(A) \# H)^m = 0$. Now $J_H(A) \# H$ is an ideal of $A \# H$, so $J_H(A) \# H \subseteq J(A \# H)$, or $J_H(A) \subseteq J(A \# H) \cap A = \mathscr{J}(A)$.

It is easy to show that for left Artinian H-module algebras, A is left H-primitive if and only if A has an irreducible left module, the annihilator of which contains no nonzero H-ideal of A. Similarly, for left Artinian H-module algebras, a left H-primitive ideal I is the largest H-ideal contained in some primitive ideal. (Statements in this paragraph do not require any of the restrictions on H.)

One would like answers to the following questions, which have been left open here:

1. When is $\mathscr{J}(A) \# H = J(A \# H)$? Theorem 6 says only that $\mathscr{J}(A) \# H \subseteq J(A \# H)$.

2. Is $J(A \# H) \subseteq J_H(A) \# H$? Again, Theorem 6 says only that $J(A \# H) \cap A \subseteq J_H(A)$.

Whereas this paper concerns itself with the abstract theory of the H-radical $\mathscr{J}$, a paper by R. E. Block [2] will give structure theorems for certain H-primitive algebras with finiteness conditions (either on the algebra or the module), carrying further the work in [1].

REFERENCES

1. R. E. BLOCK, Determination of the differentiably simple rings with a minimal ideal, *Ann. of Math.* **90** (1969), 433–459.

2. R. E. Bloch, Modules over differential polynomial rings, *Bull. Amer. Math. Soc.* **79** (1973), 729–733.
3. N. Bourbaki, "Groupes et Algèbres de Lie," Chap. 2, Fasc. 37, Hermann, Paris, 1972.
4. N. J. Divinsky, "Rings and Radicals," University of Toronto Press, Toronto, Ontario, 1965.
5. J. R. Risher, Radicals for Hopf-module algebras, Ph.D. Thesis, University of California, Riverside, December, 1971.
6. M. E. Sweedler, "Hopf Algebras," Benjamin, New York, 1969.