# Category Isotypes

typs: Cat → Cat

**C** ↦ typs(**C**)

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# 1. Basic definitions

For X an object in category C, define the collection of C-objects isomorphic to object X

$$\mathrm{typ}(X) \overset{\text{def}}{=}{}^{1} \{ Y \mid Y \cong X \}$$

- for all objects X of C, we have $X \in \mathrm{typ}(X)$.
- in general, $Y \in \mathrm{typ}(X)$ might arise via various isomorphisims $Y \cong X$.

# **derived isotype category typs(C)**

The objects of derived category typs(**C**) are the `typ(X)` *collections* for `X` an object of **C**.

The morphisms are defined next …

# maps for typs(C)

Assume typ(U)=typ(X) and typ(V)=typ(Y). In the following map box diagram the verticle maps ⇅ represents any relevant isomorphisms for U≅X and V≅Y.

Now any map X→Y corresponds to a map U→V and vice versa. Similarly, diagonal maps X→V, U→Y lift up or push down in the C-map box diagram (D).

```
X  →  Y
⇅  ↘↗ ⇅            (D)
U  →  V
```

This diagram describes essential map conversions in C where

X≅U and Y≅V.

# hom(typ(X),typ(Y)) for typs(**C**)

$\mathrm{hom}(\mathrm{typ}(X),\mathrm{typ}(Y)) \overset{\mathrm{def}}{\equiv}{}^{2}$

$\{\ F \mid F{:}\{A \mid A\cong X\} \to \{B \mid B\cong Y\}\ \} =$

$\{\ A\to B \mid A\cong X \wedge B\cong Y\ \}$

$\mathrm{id}(\mathrm{typ}(X)){:}\ \mathrm{typ}(X) \to \mathrm{typ}(X)$

$\mathrm{id}(\mathrm{typ}(X))(A) \overset{\mathrm{def}}{\equiv}{}^{2}\ A \quad \text{for } A\in\mathrm{typ}(X)$

$\mathrm{id}_{\mathrm{typ}(X)}(A)=A$

# composition of maps in typs(C)

How is the composition of morphisms in typs(C) characterized?

```
H = hom(typ(X),typ(Y)) ∘ hom(typ(Y),typ(Z))
```

The following box diagram is relevant, for any U≅X, V≅Y and W≅Z

```
        a       b

X   →   Y   →   Z

⇅  ↘↗   ⇅  ↘↗   ⇅        (DD)

U   →   V   →   W
```

```
     a     b
X  →  Y  →  Z
⇅ ↘↗ ⇅ ↘↗ ⇅        (DD)
U  →  V  →  W
      |
   “hinge”
```

This diagram is related to diagram (D) above: Take two copies of (D), one corresponding to X→Y and the other corresponding to Y→Z and hinge them together.

The composition map should be the collection of all **C**-map compositions travelling from the left side of the diagram (DD) to the right side of the diagram. That is, H is all morphisms in hom(typ(X),typ(Z)) which factor through some V, V≅Y (the middle column — "hinge" — of DD).

$$H = hom(typ(X),typ(Y)) \circ hom(typ(Y),typ(Z))$$

$$H \stackrel{def}{=}{}^{3} \{ b \circ a \mid a{:}U \to V \wedge b{:}V \to W \wedge U \cong X \wedge V \cong Y \wedge W \cong Z \}$$

Notice that H includes all the map paths from left edge to right edge in diagram (DD) that match in the "hinge". For example, X→V→Z is obtained for case U=X and W=Z.

This concludes the basic description of objects and morphisms for the derived category `typs(`**C**`)` of *isotypes* for a category **C**.

The use of collections (sets or classes) of **C**-objects and **C**-morphisms is essential for the constructions of `typs(`**C**`)` objects and morphisms.

# 2. Isovalence

We demonstrate that for a category **C**, two **C**-isotypes in `typs(`**C**`)` are isomorphic if, and only if they are actually equal.

***ISOVALENCE THEOREM***. Using the box hinge constructions

`typ(X)` ≅ `typ(Y)` in category typs(**C**) implies that `typ(X)=typ(Y)`.

… for the proof we use an informal *shorthand* notation for (collection/box) isotypes

`{X} = typ(X)`

`{X→Y} = hom(typ(X),typ(Y))`

where `X` and `Y` are objects of category `C` proof. Suppose that $\{X\} \cong \{Y\}$ …

$$
\begin{array}{c}
\varphi\rightarrow \\
\{X\} \rightleftarrows \{Y\} \\
\leftarrow\psi
\end{array}
$$

$$\psi \circ \varphi = id(\{X\})$$

$$\varphi \circ \psi = id(\{Y\})$$

So there are C-morphisms $a$ and $b$ which have matching objects at the hinges, for which $A \cong X$ and $B \cong Y$ in such a way that

$$A \xrightarrow{a} B \xrightarrow{b} A \qquad b \circ a = id(A)$$

$$B \xrightarrow{b} A \xrightarrow{a} B \qquad a \circ b = id(B)$$

This means that (*match at hinge*)

$$A \cong B \text{ and } X \cong Y$$

and thus

$$\mathrm{typ}(X)=\{X\}=\{Y\}=\mathrm{typ}(Y)$$

We are using an explicit assumption that two collections are equal if and only if they have the same members.)

QED

# 3. Implicit universe

Suppose that **C** is the category whose isotypes we wish to investigate

typs(**C**) ={ typ(X) | X object_of **C** }

We would say that in this context that *the category* **C** *determines a **relevant universe** for* **C***'s isotypes*.

For example, if **L** were a category of lattices (suitably and explicitly formulated) then typs(**L**) would be the isotypes for that universe of lattices **L**.

However, categories of lattices can be explicitly formulated in many suitable ways. For example …

***Lattices as partially ordered sets*** $(L, \leq)$ requires a set of objects L ordered by ≤ in such a way that any subset {a,b} of L has a least upper bound $a \vee b$ and a greatest lower bound $a \wedge b$. A category ***Lo*** can be specified whose objects are the elements of L and whose morphisms are the functions $F: L \to L$ which preserve the order of the elements:

$$F: L \to L,\ X \in L,\ Y \in L,\ X \leq Y \Rightarrow F(X) \leq F(Y).$$

***Lattices as algebraic variety structures on sets*** $(L, \vee, \wedge)$ requires operators to satisfy equations (*absorption laws*)

$$X \vee (X \wedge Y) = X$$

$$X \wedge (X \vee Y) = X$$

This gives a category ***La*** whose objects are the elements of L and whose morphisms are the functions $F: L \to L$ between satisfy preserve the operators

$$F(X \wedge Y) = F(X) \wedge F(Y)$$

$$F(X \vee Y) = F(X) \vee F(Y)$$

# `typs`(***Lo***) vs. `typs`(***La***)

Both categories ***Lo** and **La*** induce isotypes. The two categories ***Lo** and **La*** are functor-equivalent via a functor ***F*** : ***Lo*** → ***La*** which sends an order lattice (L, ≤) to the corresponding algebraic lattice (L,∨,∧), and a reverse functor ***G*** : ***La*** → ***Lo***, in such a way that ***F***∘***G*** is the identity functor on ***Lo*** and ***G***∘***F*** is the identity functor on ***La.***

***Lo*** ⇄***La.*** (via isofunctors )

The isofunctors ***F*** and ***G*** force derived isofunctors for `typs`(***Lo***) and `typs`(***La***), meaning that they are *equivalent* via isofunctors

`typs`(***Lo***) ⇄ `typs`(***La***). (via induced isofunctors)

over the different universes ***Lo*** and ***La***. (I have hand-waved differences in operation signatures for lattices.)

These meta-isotypes `typs`(***Lo***) and `typs`(***La***) are functor equivalent but corresponding isotypes are not equal. They are *different types of lattice types*, so to speak.

# 4. Varietal Isotypes

An interesting and challenging idea for types of types are the isotypes generated by an algebraic variety (universal algebra) **V**. ***La*** discussed in §3 is an example.

When a finitary algebraic variety **V** determines the category universe **C** for isotypes we also refer to the corresponding variety **V** as the universe, typs(**C**)=typs(**V**).

The WikipediA page for Variety (universal algebra) has a concise brief outline regarding the definition of finitary algebraic categories associated with a variety of algebras and the category monads associated with them.

# 5. Questions

A. Are types 0, 1, and 2 isotypes ?

B. Type constructors +, ×, →, Σ, Π using isotypes ?

C. Are types(V) determined by subvarieties of V ?

… to be continued … these constructions need to be relevant to category isotype theory for varieties, and intuitively compatible with constructive type theory …

Exercises

To be continued.